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THE EDGE DENSITY OF 4-CRITICAL PLANAR GRAPHS

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Several constructions of 4-critical planar graphs are given. These provide answers to two questions of B. Grünbaum and give improved bounds for the maximum edge density of such graphs.

If G is a graph we denote by $v(G)$, $e(G)$ and $\delta(G)$ the number of vertices, number of edges and minimal valence of G, respectively. G is said to be 4-*critical* if its chromatic number is 4, but every proper subgraph of G is 3-colorable. G. A. Dirac (see [2]) conjectured that if G is a 4-critical planar graph then $\delta(G) = 3$ and T. Gallai [2] made the stronger conjecture that such a graph satisfies $e(G) \le 2v(G) - 2$. Both of these conjectures were shown to be false by G. Koester [5] who found a 4-critical 4-valent planar graph with 40 vertices, and with 80 edges. Koester's graph is shown in Figure 1.

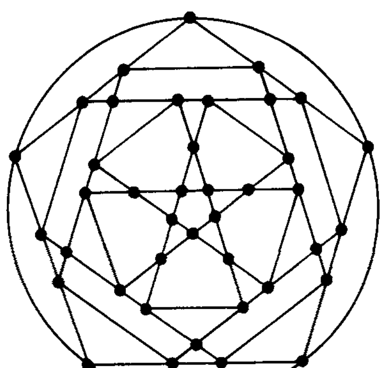

Fig. 1

B. Grünbaum [3] defined

$$S = \sup e(G)/v(G) \text{ and } L = \limsup e(G)/v(G)$$

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where the bounds are taken over all 4-critical planar graphs G. Koester's example shows $S \geq 2$. In [3] Grünbaum showed, using a construction of Gy. Hajós [4], that the following result holds:

Theorem 1. (Grünbaum) $L = S \geq 79/39 = 2.02564\ldots$

At the end of his paper, Grünbaum posed the following problems:

(i) Obtain improved lower bounds for S and improve the trivial upper bound $S \leq 3$ obtainable from Euler's formula.

(ii) Does every 4-critical planar graph G satisfy $\delta(G) \leq 4$?

(iii) Do there exist 3-connected 4-critical planar graphs G satisfying $e(G)/v(G) > 2$? (The construction establishing Theorem 1 produces graphs which are 2-connected, but not 3-connected.)

(iv) Do there exist arbitrarily large 4-valent 4-critical graphs?

In this note we provide affirmative answers to questions (iii) (Theorem 5) and (iv) (Theorem 6) and provide some new information concerning (i) (Theorem 2 and 3).

Theorem 2. $S \geq 39/19 = 2.05263\ldots$

Proof. Our argument is based on a variant of Hajós's construction, due to G. A. Dirac [1]. Let G and G' be 4-critical planar graphs. Let a, b and c be consecutive vertices on the infinite face of G and let a', b' and c' be consecutive vertices on the infinite face of G'. Let H be the graph obtained by identifying a with a', b with b', deleting the edges bc and $b'c'$ and adding the edge cc'. See Figure 2.

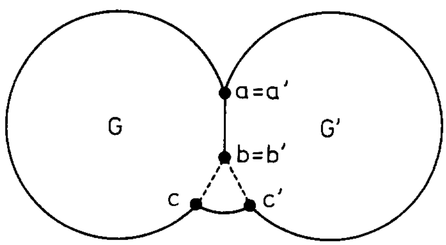

Fig. 2

H is clearly planar. Also if G and G' have minimal valency 4 and satisfy $e/v > 2$, H also has these properties. We now show that H is 4-critical. It is clearly 4-chromatic. Let E be an edge of H. We need to exhibit a 3-coloring of $H - E$. If $E = cc'$, 3-color $G - bc$ and $G' - b'c'$ so that b, c, b', c' are colored red and a and a' are colored blue. This gives a 3-coloring of $H - E$. If $E = ab$, 3-color $G - ab$ so that a and b are red and c is blue. 3-color $G' - a'b'$ so that a' and b' are red and c' is green. This induces a 3-coloring of $H - E$. If E is an edge of $G - ab$, 3-color $G' - b'c'$ so that b' and c' are red and a' is blue and 3-color $G - E$ so that b is red and a is blue. Observe that c is not red. Thus we get a 3-coloring of $G - E$. The same argument works if E is an edge of $G' - a'b'$.

Note that H has $v(G)+v(G')-2$ vertices and $e(G)+e(G')-2$ edges. Starting with Koester's graph and applying the above construction repeatedly will yield graphs with $19\cdot 2^t+2$ vertices and $39\cdot 2^t+2$ edges for any positive integer t. Thus $S \geq 39/19$.∎

Remark: Dirac's construction also preserves 3-connectivity. Thus we get an affirmative answer to Grünbaum's question (iii). However, a sharper result will be shown later.

Theorem 3. $S \leq 11/4 = 2.75$.

Proof. The key idea is the following: If G is a 4-critical planar graph and if G is not a wheel, then every vertex is incident with a face with at least four edges. To see this suppose that some vertex v of a 4-critical planar graph G is incident only with triangular faces. Let W be the wheel with hub v. W cannot have an odd number of spokes; if it did, W itself would be 4-critical and this would imply $G = W$. Delete an edge E of the rim of W. In any 3-coloring of $G - E$ the vertices of the rim of W must be colored using only two colors, and thus the end points of E must be assigned opposite colors. But this implies that G itself is 3-colorable.

Let F denote the number of faces of a 4-critical planar graph G and let F_i denote the number of faces with i edges. Then, writing $e = e(G)$ and $v = v(G)$, we get

(1) $$2e = 3\Big(F - \sum_{i\geq 4} F_i\Big) + \sum_{i\geq 4} iF_i.$$

From the observation in the preceding paragraph, $\sum_{i\geq 4} iF_i \geq v$, so that

$$2e \geq 3\Big(F - \sum_{i\geq 4} F_i\Big) + v.$$

Define k by

$$\sum_{i\geq 4} F_i = kv.$$

Thus

$$2e \geq 3(F - kv) + v$$

so that

$$F \leq \frac{2e + (3k-1)v}{3}.$$

From Euler's formula, $v + F - e = 2$, it follows that

$$v + \frac{2e + (3k-1)v}{3} - e \geq 2$$

from which we get

(2) $$\frac{e}{v} \leq 3k + 2 - \frac{6}{v}.$$

Define t by

$$\sum_{i\geq 4} iF_i = t\sum_{i\geq 4} F_i = tkv,$$

and note that $t \geq 4$. It follows from (1) that

$$2e = 3(F - kv) + tkv,$$

so that

$$F = \frac{1}{3}\{2e - (t-3)kv)\}.$$

This with Euler's formula gives

$$v + \frac{1}{3}\{2e - (t-3)kv\} - e = 2$$

from which it follows that

$$\frac{e}{v} = 3 - (t-3)k - \frac{6}{v} \leq 3 - k - \frac{6}{v}. \tag{3}$$

Thus from (2) and (3) we get

$$\frac{e}{v} \leq \max_k \min\left\{3k + 2 - \frac{6}{v}, 3 - k - \frac{6}{v}\right\} \qquad \text{and thus} \qquad S \leq \frac{11}{4}. \qquad \blacksquare$$

Lemma. *Let G be a 4-critical planar graph with at least one triangular face T with vertices a, b and c, say. Let H be the graph obtained from G by subdividing the edges ab, bc and ca by inserting new vertices x, y and z and adding the edges xy, xz and yz as in Figure 3. Then H is a 4-critical planar graph.*

Proof. H is clearly planar. We show that H is 4-critical. Let E be an edge of H. We need to exhibit a 3-coloring of $H - E$. If $E = ax$, 3-color $G - ac$ so that a and c are red and b is blue. Color x red, y green and z blue. This is a 3-coloring of $H - E$. If $E = xy$, 3-color $G - ac$ so that a and c are red and b is blue. Color x and y green and z blue. This is a 3-coloring of $H - E$. If E is an edge of $G - \{ab, bc, ac\}$, then 3-color $G - E$ so that a is red, b is blue and c is green. Color x green, y red and z blue. This is again a 3-coloring of $H - E$. ∎

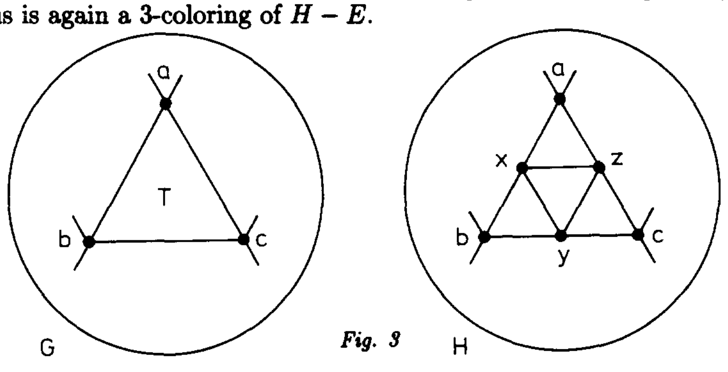

Fig. 3

In what follows we denote Koester's graph by K.

Theorem 4. *For each $n \geq 78$ there exists a 4-critical planar graph G of order n satisfying $\delta(G) \geq 4$ and $e(G)/v(G) > 2$.*

Proof. If there is such a graph G of order n and if G has a triangular face, then the graph H obtained from G via the construction in the lemma is such a graph of order $n + 3$, and it too has a triangular face. It therefore suffices to exhibit such graphs of order 78, 79 and 80. If we apply Dirac's construction to two copies of K we get such a graph of order 78. A graph of order 79 with the desired properties may be obtained by applying the Hajós construction to two copies of K, as was done by Grünbaum. To get such a graph of order 80 we proceed as follows: Apply Dirac's construction to

K and K_4. The resulting graph H has 42 vertices, 84 edges and all vertices except two have degree four. The two exceptional vertices a and b are adjacent and have degree three. Apply Dirac's construction to H and K in such a way that a and b play the roles of the vertices labelled such in Figure 2. The resulting graph G of order 80 satisfies $\delta(G) \geq 4$ and $e(G)/v(G) > 2$. ∎

Theorem 5. *For each* $n \geq 116$ *there exists a 3-connected 4-critical planar graph* G *of order* n *satisfying* $\delta(G) \geq 4$ *and* $e(G)/v(G) > 2$.

Proof. The construction given in the lemma preserves 3-connectivity. Thus it suffices to exhibit such graphs of order 116, 117 and 118. Let G be the 78-point graph obtained by applying Dirac's construction to two copies of K. The graph obtained by applying Dirac's construction to G and K is then a 116-point graph with the desired properties. Let H be a 3-connected 4-valent 4-critical planar graph of order 79. That such a graph exists follows from the lemma (see Theorem 6). A graph of order 117 is then obtained by applying Dirac's construction to H and K. The 80-point graph G constructed in the proof of Theorem 4 is 3-connected. If we apply Dirac's construction to G and K we get a graph of order 118 with all of the desired properties. ∎

Theorem 6. *For each* $n \geq 40$, $n \equiv 1 \pmod 3$, *there exists a 4-valent 4-critical planar graph of order* n.

Proof. Koester's graph is such a graph of order 40. If, in the lemma, the graph G is 4-valent and of order n then H is also 4-valent and is of order $n+3$. ∎

We do not know whether there exists 4-valent 4-critical planar graphs of order n for all sufficiently large n. We can show that if there exist such graphs of order a and b then there exists such a graph of order $a+b+2$. Thus it would suffice to find one such graph whose order a satisfies $a \not\equiv 1 \pmod 3$.

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